LAPLACIANS ON SHIFTED MULTICOMPLEXES

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Abstract. We define the Laplacian operator on finite multicomplexes and give a formula for its spectra in the case of shifted multicomplexes.

1. Introduction

The Laplacian of an undirected graph is a square matrix, whose eigenvalues yield important information. We can regard graphs as one-dimensional simplicial complexes, and ask whether there is a generalisation of the Laplacian operator to simplicial complexes. It turns out that there is, and that is useful for calculating real Betti numbers [8].

Duval and Reiner [5] have studied Laplacians of a special class of simplicial complexes, the so called *shifted* simplicial complexes. They show that such Laplacians have integral spectra, computable by a simple combinatorial formula.

Snellman [11] studied a family of monomial algebras, corresponding to truncations of the ring of arithmetical functions with Dirichlet convolution. The defining monomial ideals turn out to be *strongly stable*, which makes it possible to use the so-called Eliahou-Kervaire resolution [7] to calculate the Betti numbers of the ideals.

Now, simplicial ideals correspond to square-free monomial ideals, or differently put, to monomial ideals in the exterior algebra. Shifted simplicial complexes correspond to strongly stable square-free ideals. Monomial ideals, on the other hand, correspond to so-called *multicomplexes*. One is naturally led to the question: is there a way to define Laplacian operators on finite multicomplexes, and if so, is there a simple formula for their spectra in the case of multicomplexes corresponding to strongly stable monomial ideals?

The Laplacian operator on simplicial complexes is defined as

$$L'_d = \partial_{d+1}\partial^*_{d+1}, \tag{1}$$

where ∂ is the simplicial boundary operator, and ∂^* is its dual. Björner and Vrećica [2] defined a boundary operator on multicomplexes which generalises the boundary operator on simplicial complexes. Using their definition, we can define Laplacians on multicomplexes in analogy with (1). It turns out that the formulas of Duval and Reiner for the spectra of Laplacians of shifted simplicial complexes generalise neatly to the case of shifted multicomplexes, that is to say, to strongly stable, artinian monomial ideals. We finally arrive at a formula for the spectrum of the Laplacian of the defining ideals of the truncation algebras studied by Snellman in [11].

2. Laplacians of multicomplexes

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set, which is totally ordered, so that

$$x_1 < x_2 < \cdots < x_n \tag{2}$$

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By a *multicomplex* on X we mean a finite subset $M \subset [X]$ of the free abelian monoid on X which is closed under taking divisors, i.e.

$$m \,|t \text{ and } t \in M \quad \implies \quad m \in M \tag{3}$$

We define the total degree of a monomial in $[X]$ by

$$|x_1^{a_1} \cdots x_n^{a_n}| = \sum_{i=1}^{n} a_i \tag{4}$$

and let $[X]_\ell$ and M_ℓ denote the subset of monomials in $[X]$ and M of total degree ℓ.

2.1. **The homology theory of Björner and Vrećica.** Björner and Vrećica [2] defined a boundary operator ∂ on $\mathbb{Z}[X]$ by

$$\begin{aligned} &\partial_d : \mathbb{Z}[X]_d \mapsto \mathbb{Z}[X]_{d-1} \\ &\partial_d(x_{i_0}^{\alpha_0} \cdots x_{i_k}^{\alpha_k}) = \sum_{j=0}^{k} (-1)^{\alpha_0 + \dots \alpha_{j-1}} \cdot r_j \cdot x_{i_0}^{\alpha_0} \cdots x_{i_j}^{\alpha_j - 1} \cdots x_{i_k}^{\alpha_k}, \end{aligned} \tag{5}$$

where $r_j = 0$ if α_j is even, and $r_j = 1$ if α_j is odd. The map ∂ satisfies $\partial^2 = 0$, and since M is closed under taking divisors, $\partial(\mathbb{Z}M) \subset \mathbb{Z}M$, so ∂ is a boundary on $\mathbb{Z}M$.

Every monomial $m \in M$ can be uniquely written

$$m = p^2 q \tag{6}$$

where q is square-free. We define

$$M^{(p^2)} = \left\{\, q \in M \middle| p^2 q \in M,\, q \text{ square-free} \,\right\} \tag{7}$$

Then $M^{(p^2)}$ is a simplicial complex, and hence the graded $\mathbb{Z}$-module $\mathbb{Z}M$ decomposes as

$$\mathbb{Z}M = \bigoplus_{p^2 \in M} p^2 \mathbb{Z} M^{(p^2)} \tag{8}$$

or, taking the grading into account,

$$\mathbb{Z}M_\ell = \bigoplus_{p^2 \in M,\, 2|p| \le \ell} p^2 \mathbb{Z} M_{\ell - 2|p|}^{(p^2)} \tag{9}$$

where

$$M_\nu^{(p^2)} = \left\{\, q \in M \middle| p^2 q \in M,\, q \text{ square-free, } |q| = \nu \,\right\} \tag{10}$$

To simplify matters, we have shifted the ordinary grading on simplicial complexes so that the empty simplex lies in dimension zero rather than in dimension -1. Defining the $\boldsymbol{f}$-vector of a multicomplex as the vector $\boldsymbol{f}(M) = (f_0, f_1, \dots)$, where $f_i = \# M_i$, we get that

$$\boldsymbol{f}(M) = \sum_{p^2 \in M} S^{2|p|} \boldsymbol{f}(M^{p^2}), \tag{11}$$

where $\boldsymbol{f}(M^{p^2})$ is the re-indexed $\boldsymbol{f}$-vector of the simplicial complex M^{p^2}, and $S : \mathbb{Z}^\infty \to \mathbb{Z}^\infty$ is the shift map

$$S((f_0, f_1, f_2, \dots)) = (0, f_0, f_1, f_2, \dots) \tag{12}$$

Since

$$\partial(p^2 q) = p^2 \partial(q) \tag{13}$$

the boundary operator on M restricts to the ordinary simplicial boundary operator on $M^{(p^2)}$. Hence, the decomposition (8) is a decomposition of differential graded $\mathbb{Z}$-modules. It follows that the homology of M splits as

$$H_\ell(M) \simeq \bigoplus_{p^2 \in M,\, 2|p| \leq \ell} H_{\ell-2|p|}(M^{(p^2)}) \tag{14}$$

2.2. **Definition of Laplacians on multicomplexes.** We now proceed to define the *Laplacian* of a multicomplex. It is convenient to work with real coefficients, so we extend scalars to $\mathbb{R}$ and regard ∂_d as (the restriction to $\mathbb{R}M$ of) a map $\partial_d : \mathbb{R}[X]_d \to \mathbb{R}[X]_{d-1}$. There is a dual map

$$\partial^*_{d+1} : \mathrm{Hom}_{\mathbb{R}}(M_d, \mathbb{R}) \to \mathrm{Hom}_{\mathbb{R}}(M_{d+1}, \mathbb{R}) \tag{15}$$

We identify $\mathrm{Hom}_{\mathbb{R}}(M_d, \mathbb{R}) \simeq M_d$, so that

$$\partial^*_{d+1} : \mathbb{R}M_d \to \mathbb{R}M_{d+1}$$

With respect to the natural basis, the matrix of ∂^*_{d+1} is just the transpose of the matrix of ∂_d. More explicitly, if $m = x_1^{a_1} \cdots x_n^{a_n}$ then

$$\begin{aligned} \partial^*_{d+1}(m) &= \sum_{j=1}^{n} (-1)^{a_1+\cdots+a_{j-1}} s_j x_j m, \\ s_j &= \begin{cases} 1 & \text{if } a_j \text{ is even and } x_j m \in M \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{16}$$

Definition 1.

$$\begin{aligned} L'_d &= \partial_{d+1}\partial^*_{d+1} \\ L''_d &= \partial^*_d \partial_d \\ L_d &= L'_d + L''_d \end{aligned} \tag{17}$$

The map L_d is the *Laplacian* of the multicomplex M.

While L_d is the object which interest us, it will for technical reasons be more convenient to study the map L'_d. As we shall see, the spectrum of L'_d determines that of L_d, and vice versa, so no information is lost by this change of focus.

Definition (1) is, *mutatis mutandis*, the definition of Laplacians for simplicial complexes used by Duval and Reiner [4, 5]. We will regard simplicial complexes as special cases of multicomplexes, namely as multicomplexes consisting only of square-free monomials. The definitions we make for multicomplexes (e.g., of the spectra of their Laplacians) will then specialise to the (well studied) case of simplicial complexes.

3. The spectrum of the Laplacian

From (13) it follows that

$$\begin{aligned} L'(p^2 q) &= p^2 L'(q) \\ L''(p^2 q) &= p^2 L''(q) \\ L(p^2 q) &= p^2 L(q) \end{aligned} \tag{18}$$

Following Duval [4, 5] we define the spectra $\boldsymbol{s}'_i$, $\boldsymbol{s}''_i$, $\boldsymbol{s}^{tot}_i$, of the selfadjoint, nonnegative definite operators L'_i, L''_i, L_i to be the multiset of their (real and and nonnegative) eigenvalues. We will identify such a multiset with its weakly decreasing rearrangement, which is a partition, and we will, unless otherwise stated, identify such partitions that differ only in the number of zero parts.

The spectrum of the Laplacian of M is the multiset sum of the spectra of the constituent simplicial complexes M^{p^2}.

Lemma 2.

$$
\begin{aligned}
\boldsymbol{s}_i^{'}(M,\partial) &= \sum_{p^2\in M,\, 2\deg(p)\leq i} \boldsymbol{s}_{i-2\deg(p)}^{'}(M^{p^2},\partial)\\
\boldsymbol{s}_i^{''}(M,\partial) &= \sum_{p^2\in M,\, 2\deg(p)\leq i} \boldsymbol{s}_{i-2\deg(p)}^{''}(M^{p^2},\partial)\\
\boldsymbol{s}_i^{tot}(M,\partial) &= \sum_{p^2\in M,\, 2\deg(p)\leq i} \boldsymbol{s}_{i-2\deg(p)}^{tot}(M^{p^2},\partial)
\end{aligned}
\tag{19}
$$

Since the spectra of the Laplacians of the constituent simplicial complexes are invariant under reordering of the vertices [5, Remark 3.2], the same holds true for multicomplex Laplacian spectra. Furthermore, the relations

$$
\begin{aligned}
\boldsymbol{s}_i^{''}(M^{p^2},\partial) &= \boldsymbol{s}_{i-1}^{'}(M^{p^2},\partial)\\
\boldsymbol{s}_i^{tot}(M^{p^2},\partial) &= \boldsymbol{s}_i^{'}(M^{p^2},\partial)\cup\boldsymbol{s}_i^{''}(M^{p^2},\partial)\\
\boldsymbol{s}_i^{'}(M^{p^2},\partial) &= \boldsymbol{s}_i^{tot}(M^{p^2},\partial)-\boldsymbol{s}_i^{''}(M^{p^2},\partial)
\end{aligned}
\tag{20}
$$

which holds for all simplicial complexes M^{p^2} according to [5], translates using Lemma 2 to

Lemma 3.

$$
\begin{aligned}
\boldsymbol{s}_i^{''}(M,\partial) &= \boldsymbol{s}_{i-1}^{'}(M,\partial)\\
\boldsymbol{s}_i^{tot}(M,\partial) &= \boldsymbol{s}_i^{'}(M,\partial)\cup\boldsymbol{s}_i^{''}(M,\partial)\\
\boldsymbol{s}_i^{'}(M,\partial) &= \boldsymbol{s}_i^{tot}(M,\partial)-\boldsymbol{s}_i^{''}(M,\partial)
\end{aligned}
\tag{21}
$$

Here, equality means equality as multisets, except that the multiplicity of the element zero may differ; $\cup$ *means union of multisets, i.e. all occurring multiplicities are added, and* $-$ *denotes multiset difference.*

The lemma shows that the spectra of the Laplacians $\{\, L_i | i \geq 0 \,\}$ is completely determined by the eigenvalues of the operators $\{\, L_i' | i \geq 0 \,\}$.

4. Shifted multicomplexes and their Laplacian spectra

We will give a combinatorial formula for the spectrum of L' for the special case of shifted multicomplexes.

Definition 4. A subcomplex $N \subseteq M$ is *shifted* (relative its support) if

$$
x_j m \in N, \quad i<j, \quad x_i \in N \qquad \Longrightarrow \qquad x_i m \in N \tag{22}
$$

Lemma 5. *If M is shifted, then so are all M_{p^2}, with the induced total ordering on the vertices in their supports.*

Proof. Let $x_j q \in M_{p^2}$, so that $x_j q p^2 \in M$ and $x_j q$ and p have disjoint support. Now, if $i<j$ and $x_i \in M_{p^2}$ then $x_i \in M$ and $x_i \not| p$. Hence, since M is shifted, $x_i q p^2 \in M$. It follows that $x_i q \in M_{p^2}$. □

Definition 6. Let $N \subseteq M$ be a multicomplex. The *degree-sequence* $\boldsymbol{d}_k$ is the sequence

$$
\boldsymbol{d}_k(N) = (d_1, d_2, d_3, \dots, d_n) \tag{23}
$$

where d_j denotes the number of monomials in N_k that are divisible by x_j.

Lemma 7. *If N is shifted then $\boldsymbol{d}_k(N)$ is weakly decreasing, i.e. a partition.*

Theorem 8 (Reiner and Duval)**.** *If N is a shifted simplicial complex, then*

$$\boldsymbol{s}_k^{'} = \boldsymbol{d}_k^T(N), \tag{24}$$

where $\boldsymbol{d}_k^T(N)$ is the conjugate partition to $\boldsymbol{d}_k(N)$.

In particular, the eigenvalues of L_d' are non-negative integers.

We conclude:

Theorem 9. *Suppose that M is shifted. Then*

$$\boldsymbol{s}_k^{'} = \sum_{p^2 \in M,\, 2\deg(p) \le k} \boldsymbol{d}_k^T(M_{p^2}) \tag{25}$$

In particular, the eigenvalues of L_d' are non-negative integers.

Equivalently: let $\underline{b}$ be 1 if b is odd, and zero otherwise, and let

$$\underline{(\alpha_1, \ldots, \alpha_n)} = (\underline{\alpha_1}, \ldots, \underline{\alpha_n}) \tag{26}$$

Then

$$\boldsymbol{s}_k^{'\,T} = \sum_{\boldsymbol{x}^{\boldsymbol{\alpha}} \in M_k} \underline{\boldsymbol{\alpha}} \tag{27}$$

Example 10. Let $M_3 = [x_1, x_2, x_3]_3$. Then the matrix of d_3, with respect to the basis of monomials of degree three and two ordered lexicographically, is $P =$

	x_1^3	$x_1^2x_2$	$x_1^2x_3$	$x_1x_2^2$	$x_1x_2x_3$	$x_1x_3^2$	x_2^3	$x_2^2x_3$	$x_2x_3^2$	x_3^3
x_1^2	1	-1	-1	0	0	0	0	0	0	0
x_1x_2	0	0	0	0	1	0	0	0	0	0
x_1x_3	0	0	0	0	-1	0	0	0	0	0
x_2^2	0	0	0	1	0	0	1	-1	0	0
x_2x_3	0	0	0	0	1	0	0	0	0	0
x_3^2	0	0	0	0	0	1	0	0	1	1

The symmetric matrix PP^* has eigenvalues $3, 3, 3, 3, 0, 0$. We have that

$$\begin{aligned}(3,3,3,3)^T &= (4,4,4)\\ &= (1,0,0) + (0,1,0) + (0,0,1) + (1,0,0) + (1,1,1) +\\ &\quad (1,0,0) + (0,1,0) + (0,0,1) + (0,1,0) + (0,0,1).\end{aligned}$$

Let now

$$M_3 = \left\{x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_2^2x_3, x_1x_2x_3, x_1^2x_3\right\} \subset [x_1, x_2, x_3]_3,$$

as in the left part of Table 1.

Now, the matrix of d_3, with respect to the basis of monomials of degree three and two ordered lexicographically, is $P =$

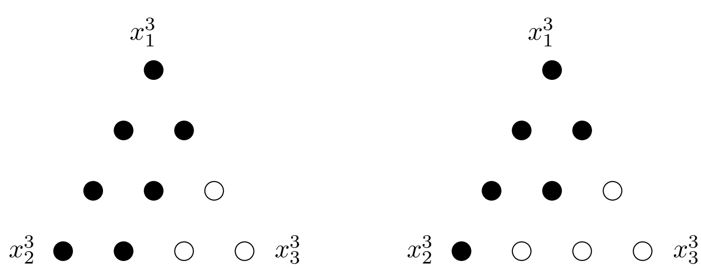

TABLE 1. Multicomplexes on three variables, degree 2

	x_1^3	$x_1^2x_2$	$x_1^2x_3$	$x_1x_2^2$	$x_1x_2x_3$	$x_1x_3^2$	x_2^3	$x_2^2x_3$	$x_2x_3^2$	x_3^3
x_1^2	1	-1	-1	0	0	x	0	0	x	x
x_1x_2	0	0	0	0	1	x	0	0	x	x
x_1x_3	0	0	0	0	-1	x	0	0	x	x
x_2^2	0	0	0	1	0	x	1	-1	x	x
x_2x_3	0	0	0	0	1	x	0	0	x	x
x_3^2	0	0	0	0	0	x	0	0	x	x

and PP^* has eigenvalues $3,3,3,0,0,0$. We have that

$$\begin{aligned}(3,3,3)^T &= (3,3,3)\\ &= (1,0,0)+(0,1,0)+(0,0,1)+(1,0,0)+(1,1,1)+\\ &\quad +(0,1,0)+(0,0,1).\end{aligned}$$

Finally, let

$$M_3 = \left\{x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_1x_2x_3, x_1^2x_3\right\} \subset [x_1,x_2,x_3]_3,$$

as in the right part of Table 1.

Then the matrix of d_3, with respect to the basis of monomials of degree three and two ordered lexicographically, is $P =$

	x_1^3	$x_1^2x_2$	$x_1^2x_3$	$x_1x_2^2$	$x_1x_2x_3$	$x_1x_3^2$	x_2^3	$x_2^2x_3$	$x_2x_3^2$	x_3^3
x_1^2	1	-1	-1	0	0	x	0	x	x	x
x_1x_2	0	0	0	0	1	x	0	x	x	x
x_1x_3	0	0	0	0	-1	x	0	x	x	x
x_2^2	0	0	0	1	0	x	1	x	x	x
x_2x_3	0	0	0	0	1	x	0	x	x	x
x_3^2	0	0	0	0	0	x	0	x	x	x

PP^* now has eigenvalues $3,3,2,0,0,0$. We have that

$$\begin{aligned}(3,3,2)^T &= (3,3,2)\\ &= (1,0,0)+(0,1,0)+(0,0,1)+(1,0,0)+(1,1,1)+(0,1,0).\end{aligned}$$

5. Strongly stable monomial ideals

Let $I \subset \mathbb{R}[x_1,\ldots,x_n]$ be an artinian monomial ideal, i.e., a monomial ideal such that the quotient ring $S = R/I$ is artinian. Let M be the set of monomials in $[x_1,\ldots,x_n]$ not in I. Then M is a finite multicomplex on $[x_1,\ldots,x_n]$.

Conversely, if M is a finite multicomplex on $[x_1,\ldots,x_n]$, then one can form the monomial ideal I generated by the monomials not in I, and M will be an $\mathbb{R}$-basis for the artinian ring $S = R/I$. We denote this ring by $\mathbb{R}M$ and call it the multicomplex ring of M. This is coherent with our previous use of $\mathbb{R}M$; we have merely introduced a multiplication on this $\mathbb{R}$-vectorspace.

By means of the correspondence $M \leftrightarrow I$ between finite multicomplexes and artinian monomial ideals, we can now define Laplacians and their spectra for artinian monomial ideals. For a special class of monomial ideals, which is important because of its connection to so-called *generic initial ideals* (see for instance [9, 6, 1]), we can use the results of the previous section to calculate these spectra.

A monomial ideal I is said to be *strongly stable* or *Borel-fixed* [9, 6] w.r.t. a total order of $X = \{x_1,\ldots,x_n\}$ if

$$m \in I \text{ and } x_i \,|m \text{ and } x_j < x_i \quad \Longrightarrow \quad \frac{x_j}{x_i} m \in I \tag{28}$$

The minimal graded free resolution of strongly stable ideals are given by the so-called Eliahou-Kervaire resolution [7]. This means that the graded Betti numbers of I, and hence of S, can be easily read of from the minimal monomial generators of I.

Lemma 11. *I is strongly stable if and only if M is shifted w.r.t. the reverse ordering on X.*

The above lemma means that the spectra of Laplacians of artinian strongly stable monomial ideals can be easily calculated from combinatorial data involving the standard monomials in the quotient ring.

6. Truncations of arithmetical functions with Dirichlet convolution

We will apply the results of section 3 to a particular family of shifted multicomplexes that arise naturally in the study of arithmetical functions. Recall that an arithmetical function is a complex-valued function defined on the positive integers. The set Γ of all such functions becomes a complex vector space under point-wise addition and multiplication by scalars. If we introduce the indicator functions e_m, defined for all positive integers m by

$$e_m(k) = \begin{cases} 1 & k = m \\ 0 & k \neq m \end{cases} \tag{29}$$

then every element $f \in \Gamma$ can be written as a formal linear combination

$$f = \sum_{m=1}^{\infty} c_m e_m, \qquad c_m \in \mathbb{C} \tag{30}$$

With respect to a natural adic topology on Γ, the above is an absolutely convergent sum.

There is also a famous convolution product, the so-called *Dirichlet convolution*, with respect to which Γ becomes an associative, commutative $\mathbb{C}$-algebra isomorphic to the unrestricted power series ring on countably many variables.

The Dirichlet convolution is defined by

$$f * g(m) = \sum_{k|m} f(k)g(m/k) \tag{31}$$

which is just the $\mathbb{C}$-linear and continuous extension of the rule

$$e_a * e_b = e_{ab} \tag{32}$$

The isomorphism between $(\Gamma, *)$ and $\mathbb{C}[[x_1, x_2, \ldots,]]$ alluded to above is given by the $\mathbb{C}$-linear and continuous extension of

$$e_{p_1^{a_1} \cdots p_r^{a_r}} \mapsto y_1^{a_1} \cdots y_r^{a_r} \tag{33}$$

where p_i is the i'th prime number. Cashwell and Everett [3] used this isomorphism to show that $(\Gamma, *)$ is a unique factorisation domain.

Now let N be a positive integer, and let Γ_N denote the subset of all arithmetical functions supported on $\{1, 2, \ldots, N\}$. With the modified multiplication

$$f *_N g(k) = \begin{cases} f * g(k) & k \leq N \\ 0 & k > N \end{cases} \tag{34}$$

Γ_N becomes a retract of Γ. Furthermore, Γ_N is the multicomplex ring on M_N, the multicomplex consisting of $\{1, 2, \ldots, N\}$, where $k = p_1^{a_1} \cdots p_r^{a_r}$ is regarded as a multiset on $\{p_1, \ldots, p_r\}$, i.e. is identified with the monomial $x_1^{a_1} \cdots x_r^{a_r}$.

In [11] the multicomplex ring Γ_N, regarded as the artinian monomial algebra $\mathbb{C}[x_1, \ldots, x_n]/I_N$, was studied. The point of departure was the fact that I_N is strongly stable w.r.t. the reverse order of the variables, so that the Eliahou-Kervaire resolution yields the graded Betti numbers of the algebra in terms of certain number-theoretic quantities. We will here use the fact that M_N is a shifted multicomplex to determine its Laplacian spectrum.

Lemma 12. *M_N is a shifted multicomplex on $y_1, \dots, y_n$, where n is the largest number so that $p_n \leq N$.*

Proof. Obvious. □

Definition 13. For a positive integer m with prime factorisation $m = p_1^{a_1} \cdots p_r^{a_r}$ we define $\log(m) = (a_1, a_2, \cdots)$. Conversely, given a finitely supported sequence $\boldsymbol{\alpha} = (a_1, a_2, a_3, \dots)$ of non-negative integers, we define $\boldsymbol{p}^{\boldsymbol{\alpha}} = p_1^{a_1} \cdots p_r^{a_r}$.

With this definition, the *squarefree part* of $\boldsymbol{p}^{\boldsymbol{\alpha}}$ is

$$\mathrm{sfp}(\boldsymbol{p}^{\boldsymbol{\alpha}}) = \boldsymbol{p}^{\underline{\boldsymbol{\alpha}}},$$

and the total degree $|\boldsymbol{\alpha}| = \sum_i a_i$ is equal to the number of not necessarily distinct prime factors of $\boldsymbol{p}^{\boldsymbol{\alpha}}$, i.e. to $\Omega(\boldsymbol{p}^{\boldsymbol{\alpha}})$ to use the notation in [10].

Corollary 14. *The spectrum $\boldsymbol{s}_k^{'}$ of the k'th Laplacian L_k' of the multicomplex M_N is given by*

$$\boldsymbol{s}_k^{'\,T} = \sum_{\substack{1 \leq \ell \leq N \\ \Omega(\ell) = k}} \underline{\log(\ell)} = \sum_{\substack{1 \leq \ell \leq N \\ \Omega(\ell) = k}} \log(\mathrm{sfp}(\ell)) \tag{35}$$

Hence, if we introduce the arithmetical functions t_k^i and s_k^i by

$$\begin{aligned} \boldsymbol{s}_k^{'}(N) &= (t_k^1(N), t_k^2(N), \dots) \\ \boldsymbol{s}_k^{'}(N)^T &= (s_k^1(N), s_k^2(N), \dots) \end{aligned} \tag{36}$$

then

$$\begin{aligned} t_k^i(N) &= \sum_{\substack{1 < n \leq N \\ \Omega(n) = k \\ p_i | \mathrm{sfp}(n)}} 1 \\ s_k^j(N) &= \sum_{\left\{ i \middle| t_k^i(N) \geq j \right\}} 1 \\ &= \#\left\{ \ell :\, \#\left\{ 1 < n \leq N :\, \Omega(n) = k,\, p_\ell \,| \mathrm{sfp}(n) \right\} \geq j \right\} \end{aligned} \tag{37}$$

We have that $s_1^1(N)$ is the number of primes $\leq N$, and that $s_1^j(N) = 0$ for $j > 1$. To study $t_2^i(N)$ and $s_2^i(N)$, first note that

$$\mathrm{sfp}(p_a p_b) = \begin{cases} p_a p_b & a \neq b \\ 1 & a = b \end{cases} \tag{38}$$

It follows that if we define

$$\begin{aligned} Y_2(N) &= (y_{ab}), \qquad y_{ab} = \begin{cases} 1 & a \neq b,\, p_a p_b \leq N \\ 0 & \text{otherwise} \end{cases} \\ U_2(N) &= (u_{ab}), \qquad u_{ab} = \begin{cases} y_{ab} & b < a \\ y_{a,b+1} & b \geq a \end{cases} \end{aligned} \tag{39}$$

then $t_2^i(N)$ is the i'th row sum of $U_2(N)$, and $s_2^j(N)$ is the j'th column sum of $U_2(N)$. An example, for $N = 50$, is shown in Table 2.

Since $p_n \sim n \log n$, we see that for fixed i and large N,

$$s_2^i(N) \sim \frac{N/p_i}{\mathcal{W}(N/p_i)} \tag{40}$$

$$\begin{bmatrix} 0&1&1&1&1&1&1&1&1\\ 1&0&1&1&1&1&0&0&0\\ 1&1&0&1&0&0&0&0&0\\ 1&1&1&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0 \end{bmatrix} \quad \begin{bmatrix} 1&1&1&1&1&1&1&1&0\\ 1&1&1&1&1&0&0&0&0\\ 1&1&1&0&0&0&0&0&0\\ 1&1&1&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0 \end{bmatrix}$$

TABLE 2. $Y_2(50)$ and $U_2(50)$. $\mathbf{t}_2 = (8,5,3,3,2,2,1,1,1)$, $\mathbf{s}_2 = (9,6,4,2,2,1,1,1)$.

where $\mathcal{W}$ is the Lambert W-function. If we allow also i to tend to infinity, but much slower than N, then we get that

$$s_2^i(N) \sim \frac{N/(i \log i)}{\mathcal{W}(N/(i \log i))} \tag{41}$$

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